Math 347 Worksheet

Lecture 6: Applications of Induction

September 14, 2018

1) Set up an appropriate induction and prove the following formulas:

   (i) For all $n \geq 1$ one has
   $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2};$$

   (ii) for all $n \geq 1$ one has
   $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (iii) (Extra) Can you find an argument for (i) without using induction? (Hint: Look at Anecdote from Carl F. Gauss, for instance on Wikipedia.)

   (iv) (Extra) Can you find a similar argument for (ii)?

2) Obtain a simple formula for the number of closed intervals with integer endpoints contained in the interval $[1, n]$ (including one-point intervals).

3) Let q be a real number other than 1. Prove the following formula
$$\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1}.$$

4) Consider a party with n married couples. People shake hands with each other, but no two spouses shake hands among themselves. If all of the $2n-1$ people except for the host shake hands with a different number of people[1]. Prove that the hostess shake hands with exactly $n-1$ persons.

5) Find the fault in the following proof and fix it.

   Claim: For all $n \geq 1$ one has $3^n \leq 3n!$[2].

   Proof: Consider $n = 1$, *one has* $3^1 = 3 \leq 3 \cdot 1! = 3$. *Now we assume that* $3^n \leq 3 \cdot n!$, *then one has*
   $$3^{n+1} = 3^n \cdot 3 \leq 3 \cdot n! \cdot 3 \leq 3 \cdot n! \cdot (n+1) = 3 \cdot (n+1)!$$
   Thus, $3^n \leq 3 \cdot n!$, *for all* $n \geq 1$.

6) For any natural number $n \geq 1$, prove that
$$|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|,$$
for any collection of real numbers $a_i \in \mathbb{R}$, for all $1 \leq i \leq n$.

[1] In other words, except for the host, from the $2n-1$ people each of them shake hands a different number of times than anyone else except for the host.

[2] Recall $n! = 1 \cdot 2 \cdot \ldots \cdot n$.